Solution of HW8

16 We use the criterion (51) and (56) in textbook.

(a) Note that

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{x!}{(x+1)!} = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty,$$

and

$$\sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \sum_{x=1}^{\infty} \frac{x!}{x!} = \sum_{x=1}^{\infty} 1 = \infty.$$

Hence the process is null recurrent.

(b) Note that

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \sum_{x=1}^{\infty} \frac{x!}{(x+2)!} = \sum_{x=1}^{\infty} \frac{1}{(x+1)(x+2)} = \sum_{x=1}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+2} \right) = \frac{1}{2} < \infty.$$

Hence the process is transient.

17 Consider the embedded Markov chain (in page 102 of textbook) with transition function

$$P(x,y) = Q_{xy} = \begin{cases} 1, & x = 0,\ y = 1; \\ \dfrac{\lambda_x}{\lambda_x + \mu_x} = p_x, & y = x+1,\ x \geq 1; \\ \dfrac{\mu_x}{\lambda_x + \mu_x} = q_x, & y = x-1,\ x \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

We see that the embedded chain is a birth and death chain on the nonnegative integers. Moreover, $\gamma_0 = 1$, and for $x \geq 1$,

$$\gamma_x = \frac{q_1 \cdots q_x}{p_1 \cdots p_x} = \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x}.$$

Using Q26 of Chapter 1, we get (a) and (b) immediately.

18 **(a)** Note that $\gamma_y = (\frac{\mu}{\lambda})^y$. As now $\mu \geq \lambda$, so $\sum_y \gamma_y = \infty$. Hence $\rho_{x0} = 1$ by Q17(a).

(b) If $\mu < \lambda$, by Q17(b),

$$\rho_{x=0} = \frac{\sum_{y=x}^{\infty} (\mu/\lambda)^y}{\sum_{y=0}^{\infty} (\mu/\lambda)^y} = \left(\frac{\mu}{\lambda} \right)^x, \quad x \geq 1.$$

19 Note that $\gamma_y = (\frac{1-p}{p})^y$. If $p \le \frac{1}{2}$, then $\sum_y \gamma_y = \infty$. Hence $\rho_{x0} = 1$ by Q17(a). If $p > \frac{1}{2}$, then by Q17(b),

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} (\frac{1-p}{p})^y}{\sum_{y=0}^{\infty} (\frac{1-p}{p})^y} = \left(\frac{1-p}{p}\right)^x, \quad x \ge 1.$$

21 Using the result in page 105 of textbook, set $\pi_0 = 1$, and

$$\pi_x = \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x, \quad 1 \le x \le d.$$

Then the stationary distribution is given by

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{d} \pi_y} = \frac{\frac{(\lambda/\mu)^x}{x!}}{\sum_{y=0}^{d} \frac{(\lambda/\mu)^y}{y!}}, \quad 0 \le x \le d.$$

SQ1 **(a)** The rate matrix is given by

$$D = \begin{array}{c} \begin{matrix} \ \ 0 & \ \ 1 & \ \ 2 & \ \ 3 \end{matrix} \\ \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}.$$

(b) Note that for birth and death process, if we put

$$\pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} \text{ for } x = 1, 2 \text{ and } 3,$$

then the stationary distribution π is given by

$$\pi(x) = \begin{cases} (1 + \sum_{y=1}^{3} \pi_y)^{-1} & \text{if } x = 0, \\ \pi_x (1 + \sum_{y=1}^{3} \pi_y)^{-1} & \text{if } x = 1, 2 \text{ or } 3. \end{cases}$$

Since $\lambda_0 = \lambda_1 = \lambda_2$ and $\mu_1 = \mu_2 = \mu_3 = 1$, it follows that

$$\pi = (1/15, 2/15, 4/15, 8/15).$$

The required probability is $\lim_{t \to \infty} P(X(t) = 2) = \pi(2) = 4/15$.

SQ2 Let $X(t)$ be the number of customers that we are waiting or being served at time t.

(a) The infinite matrix is given by

$$\begin{array}{c} \begin{matrix} \;\;0 & \;\;1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 & \cdots \end{matrix} \\ D = \begin{bmatrix} -2 & 2 & & & & & \\ 2 & -4 & 2 & & & & \\ & 2 & -4 & 2 & & & \\ & & 3 & -5 & 2 & & \\ & & & 3 & -5 & 2 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}. \end{array}$$

(b) Note that $\lambda_x = 2$ for all $x \geq 0$ and

$$\mu_x = \begin{cases} 2 & \text{if } 1 \leq x \leq 2, \\ 3 & \text{if } x \geq 3. \end{cases}$$

So,

$$\pi_x = \begin{cases} 1 & \text{if } 1 \leq x \leq 2 \\ (\frac{2}{3})^{x-2} & \text{if } x \geq 3. \end{cases}$$

Let π be the stationary distribution of this birth and death process. Since $\sum_{y=1}^{\infty} \pi_y = 4$, we have

$$\pi(x) = \begin{cases} \frac{1}{5} & \text{if } 0 \leq x \leq 2, \\ \frac{1}{5}(\frac{2}{3})^{x-2} & \text{if } x \geq 3. \end{cases}$$

(c)

$$\lim_{t \to \infty} P(X(t) = 4) = \pi(4) = \frac{4}{45}.$$